REVISION QUESTIONS 1

Time: $50 \sim 55$ min, Due: Sat, 25 Dec

1. The equation $x^5 - 3x^3 + x^2 - 4 = 0$ has one positive root.

   (a) Verify by calculation that this root lies between 1 and 2. [2]

   (b) Show that the equation can be rearranged in the form [1]

   $$x = \sqrt[3]{\left(3x + \frac{4}{x^2} - 1\right)}.$$

   (c) Use an interative formula based on this rearrangement to determine the positive root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

2. Let $I = \displaystyle\int_0^1 \frac{9}{(3+x^2)^2}\,\mathrm{d}x$.

   (a) Using the substitution $x = (\sqrt{3})\tan\theta$, show that $I = \sqrt{3}\displaystyle\int_0^{\frac{1}{6}\pi} \cos^2\theta\,\mathrm{d}\theta$. [3]

   (b) Hence find the exact value of I. [4]

3. A curve has equation

   $$\sin y \ln x = x - 2\sin y,$$

   for $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$.

   (a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [5]

   (b) Henc find the exact x -coordinate of the point on the curve at which the tangent is parallel to the x -axis. [3]

4. The variables x and y satisfy the differential equation

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{x+y}$$

   and it is given that $y = 0$ when $x = 0$.

   (a) Solve the differential equation and obtain an expression for y in terms of x. [7]

   (b) Explain briefly why x can only take values less than 1. [1]

5. Let $\mathrm{f}(x) = \dfrac{3x^3 + 6x - 8}{x(x^2+2)}$.

   (a) Express $\mathrm{f}(x)$ in the form $A + \dfrac{B}{x} + \dfrac{Cx + D}{x^2+2}$. [5]

   (b) Show that $\displaystyle\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = 3 - \ln 4$. [5]

REVISION QUESTIONS 2

Time: $35 \sim 40$ min, Due: Mon, 27 Dec

1. Sketch the graph of $y = \sec x$, for $0 \leqslant x \leqslant 2\pi$. [3]

2. (a) Show that if $y = 2^x$, then the equation
$$2^x - 2^{-x} = 1$$
can be written as a quadratic equation in y. [2]

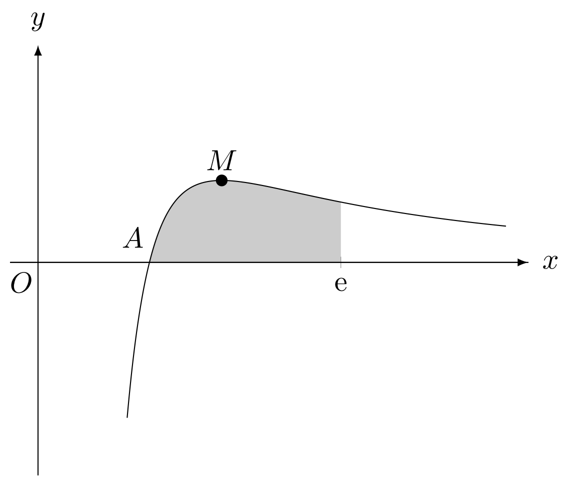
   (b) Hence solve the equation [4]
$$2^x - 2^{-x} = 1$$

3. (a) Prove the identity [3]
$$\sin^2\theta\cos^2\theta \equiv \frac{1}{8}(1 - \cos 4\theta).$$

   (b) Hence find the exact value of [3]
$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta \, \mathrm{d}\theta.$$

4. Given that $y = 1$ when $x = 0$, solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3 + 1}{y^2},$$
obtaining an expression for y in terms of x. [6]

5. The diagram shows the curve $y = \dfrac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x -axis at A.



   (a) Write down the x-coordinates of A. [1]

   (b) Find the exact coordinates of M. [5]

   (c) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

REVISION QUESTIONS 3

Time: $35 \sim 40$ min, Due: Wed, 29 Dec

1. Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

2. (a) Use the substitution $x = \tan\theta$ to show that [4]

$$\int \frac{1-x^2}{(1+x^2)^2}\,\mathrm{d}x = \int \cos 2\theta\,\mathrm{d}\theta.$$

   (b) Hence find the value of [3]

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,\mathrm{d}x.$$

3. (a) Using partial fractions, find [4]

$$\int \frac{1}{y(4-y)}\,\mathrm{d}y.$$

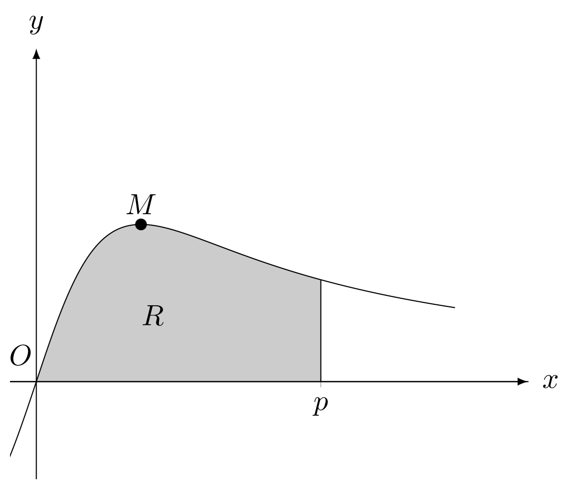
   (b) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(4-y),$$

   obtaining an expression for y in terms of x. [4]

   (c) State what happens to the value of y if x becomes very large and positive. [1]

4. The diagram shows part of the curve $y = \dfrac{x}{x^2+1}$ and its maximum point M. The shaded region R is bounded by the curve and by the lines $y = 0$ and $x = p$.



   (a) Calculate the x-coordinate of M. [4]

   (b) Find the area of R in terms of p. [3]

   (c) Hence calculate the value of p for which the area of R is 1, giving your answer correct to 3 significant figures. [2]

REVISION QUESTIONS 4

Time: $45 \sim 50$ min, Due: Fri, 31 Dec

1. Find the exact value of $\displaystyle\int_0^1 xe^{2x}\,\mathrm{d}x$. [4]

2. With respect to the origin O, the points A and B have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \overrightarrow{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$$

   The line l has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

   (a) Prove that the line l does not intersect the line through A and B. [5]

3. Let $\mathrm{f}(x) = \dfrac{9x^2 + 4}{(2x+1)(x-2)^2}$.

   (a) Express $\mathrm{f}(x)$ in partial fractions. [5]

   (b) Show that, when x is sufficiently small for x^3 and higher powers to be neglected, [4]

$$\mathrm{f}(x) = 1 - x + 5x^2.$$

4. In a chemical reaction a compound X is formed from a compound Y. The masses in grams of X and Y present at time t seconds after the start of the reaction are x and y respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of X is proportional to the mass of Y at that time. When $t = 0$, $x = 5$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1.9$.

   (a) Show that x satisfies the differential equation [2]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.02(100 - x).$$

   (b) Solve this differential equation, obtaining an expression for x in terms of t. [6]

   (c) State what happens to the value of x as t becomes very large. [1]

5. The equation of a curve is $y = \ln x + \dfrac{2}{x}$, where $x > 0$.

   (a) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

   (b) The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3 - \ln x_n}.$$

   with initial value $x_1 = 1$, converges to α. State an equation satisfied by α, and hence show that α is the x-coordinate of a point on the curve where $y = 3$. [2]

   (c) Use this iterative formula to find α correct to 2 decimal places, showing the result of each iteration. [3]

REVISION QUESTIONS 5

Time: $70 \sim 80$ min, Due: Sun, 2 Jan

1. Given that $x = 4\left(3^{-y}\right)$, express y in terms of x. [3]

2. Solve the inequality $2x > |x - 1|$. [4]

3. The parametric equations of a curve arc

$$x = 2\theta + \sin 2\theta, \qquad y = 1 - \cos 2\theta.$$

   Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$. [5]

4. (a) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

   (b) Hence solve the equation

$$7\cos\theta + 24\sin\theta = 15,$$

   giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [4]

5. In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 75$.

   (a) Show that x and t satisfy the differential equation [2]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.1(x - 250).$$

   (b) Solve this differential equation, obtaining an expression for x in terms of t. [6]

6. (a) By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + \mathrm{e}^{x},$$

   where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

   (b) Verify by calculation that this root lies between 0.5 and 1.0. [2]

   (c) Show that this root also satisfies the equation [1]

$$x = \tan^{-1}\left(\frac{2}{1 + \mathrm{e}^{x}}\right).$$

   (d) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1 + \mathrm{e}^{x_n}}\right).$$

   with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

7. The points A and B have position vectors, relative to the origin O, given by

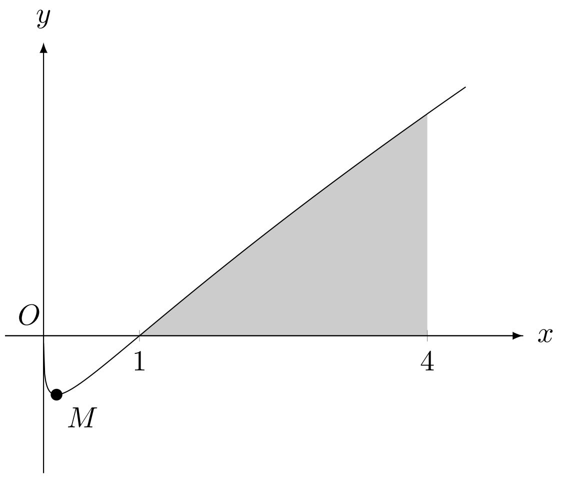
$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(a) State a vector equation for the line l. [1]

(b) Find the position vector of N and show that $BN = 3$. [6]

8. The diagram shows a sketch of the curve $y = x^{\frac{1}{2}} \ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.



(a) Find the exact value of the $x-$ coordinate of M. [4]

(b) Use integration by parts to find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

9. (a) Express $\dfrac{10}{(2-x)(1+x^2)}$ in partial fractions. [5]

(b) Hence, given that $|x| < 1$, obtain the expansion of $\dfrac{10}{(2-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [5]

REVISION QUESTIONS 6

Time: $90 \sim 100$ min, Due: Tue, 4 Jan

1. Expand $(2+3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

2. The polynomial $x^3 - 2x + a$, where a is a constant, is denoted by $p(x)$. It is given that $(x+2)$ is a factor of $p(x)$.

   (a) Find the value of a. [2]

   (b) When a has this value, find the quadratic factor of $p(x)$. [2]

3. The equation of a curve is $y = x\sin 2x$, where x is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

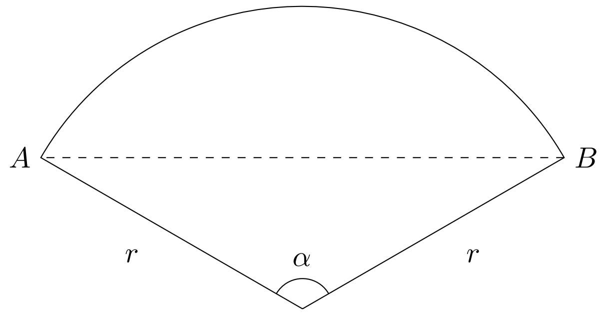
4. Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation [6]

$$3^x = 2 + 3^{-x}.$$

5. (a) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(b) Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{\left(\cos\theta + (\sqrt{3}\sin\theta)\right)^2}\,\mathrm{d}\theta = \frac{1}{\sqrt{3}}$. [4]

6. The diagram shows a sector AOB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \pi$. The area of triangle AOB is half the area of the sector.



(a) Show that α satisfies the equation [2]

$$x = 2\sin x.$$

(b) Verify by calculation that α lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. [2]

(c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\sin x_n\right)$$

converges, then it converges to a root of the equation in part (a). [2]

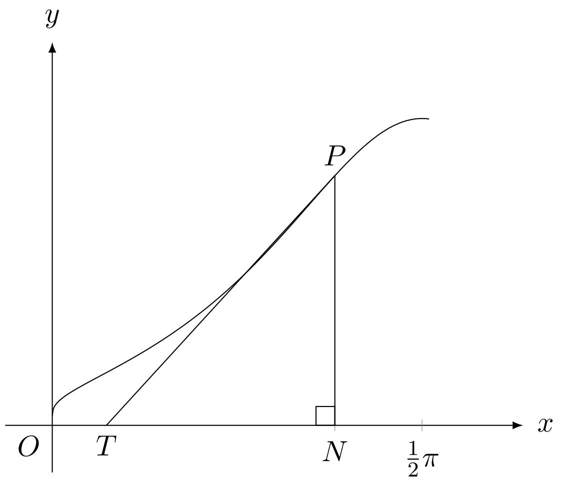
(d) Use this iterative formula, with initial value $x_1 = 1.8$, to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

7. Let $\mathrm{f}(x) \equiv \dfrac{x^2+3x+3}{(x+1)(x+3)}$.

   (a) Express $\mathrm{f}(x)$ in partial fractions. [5]

   (b) Hence show that $\displaystyle\int_0^3 \mathrm{f}(x)\,\mathrm{d}x = 3 - \frac{1}{2}\ln 2$. [4]

8. In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

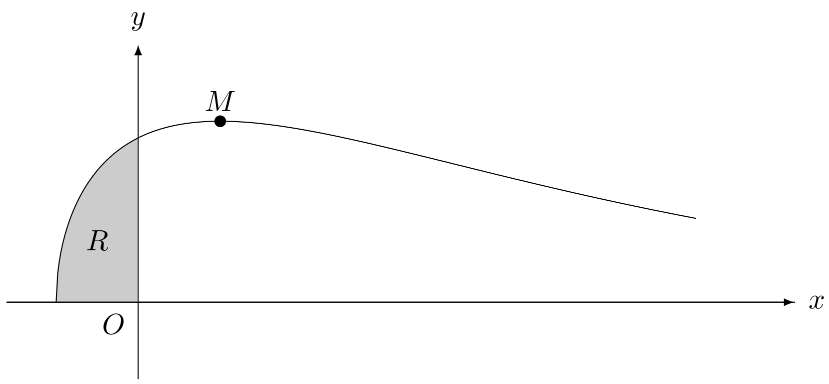


   (a) Using the fact that the gradient of the curve at P is $\dfrac{PN}{TN}$, show that [3]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}y^2 \cot x.$$

   (b) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

9. The diagram shows the curve $y = \mathrm{e}^{-\frac{1}{2}x}\sqrt{1+2x}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.



   (a) Find the x-coordinate of M. [4]

   (b) Find by integration the volum of the solid obtained when R is rotated completely about the x-axis. Give your answer in terms of π and e. [6]

10. The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation

$$\mathbf{r} = (1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}.$$

   (a) Show that l does not intersect the line passing through A and B. [4]

   (b) The point P lies on l and is such that angle PAB is equal to $60°$. Given that the position vector of P is $(1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of P. [6]